Exploring Euclid's Orchard

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Abstract

In this paper we look at integer sequences formed from enumerating lattice paths using subsets of the generalised Euclid's orchard as step sets, possibly using various different constraints.

Keywords: Lattice Paths, Enumeration, Euclid's Orchard.

MSC: 00A08, 05A15, 05C38, 05C12, 20M14, 05C20, 05C30.

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1 Introduction

2 Background

By *the integer lattice* we mean the Cartesian product of the integers with itself, ie. $\mathbb{Z}^2$. We shall refer to points of the integer lattice as *integer lattice points*.

Need to introduce lattice paths (can introduce important stuff while referring the reader to JENH paper).

Two integers $x, y \in \mathbb{Z}$ are called *coprime*, *mutually prime* or *relatively prime* if $\gcd(x, y) = 1$, which (is this true??), excluding $x = 0$ and $y = 0$, by definition occurs if and only if $\frac{x}{y}$ and $\frac{y}{x}$ are reduced.

Euclid's orchard is the subset $\{(x, y) \in \mathbb{P}^2 : x \text{ and } y \text{ are coprime}\}$, which we denote as EO. Intuitively Euclid's orchard can be thought of as the positive integer lattice points $(x, y) \in \mathbb{P}^2$ such that no other integer lattice points lie on the straight line between the origin and (x, y). Even more intuitively, treat each integer lattice point to be a tree, and if there is a tree T_2 sitting on the straight line between the origin and another tree T_1 then we shall consider tree T_1 as not visible from the origin. Euclid's orchard is the subset of trees from $\mathbb{P}^2$ that are visible from the origin, or equivalently the intersection of $\mathbb{P}^2$ with the trees that are visible from the origin.

A generalisation of Euclid's orchard is to consider the subset $\{(x, y) \in \mathbb{Z}^2 : x \text{ and } y \text{ are coprime}\}$, we shall denote this set as GEO and refer to it as *the generalised Euclid's orchard*. Again, intuitively the generalised Euclid's orchard can be thought of as the integer lattice points $(x, y) \in \mathbb{Z}^2$ such that no other integer lattice points lie on the straight line between the origin and (x, y). Also again, more intuitively the generalised Euclid's orchard is the subset of trees (from $\mathbb{Z}^2$) that are visible from the origin.

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3 n-Dimensions

Note that Euclid's orchard when generalised to $\mathbb{Q}^2$ is empty (Proof: Let $a, b, c, d \in \mathbb{Z}$ and consider $(\frac{a}{b}, \frac{c}{d}) \in \mathbb{Q}^2$, which is an arbitrary element of $\mathbb{Q}^2$. Let $y(x) = \frac{bc}{ad}x$, so $y(0) = 0$ and $y(\frac{a}{b}) = \frac{c}{d}$. Note $\mathbb{Q} \cap (0, \left|\frac{a}{b}\right|)$ is non-empty since $\mathbb{Q}$ is dense in $\mathbb{R}$. Pick any point $q \in \mathbb{Q} \cap (0, \left|\frac{a}{b}\right|)$, then $(q, y(q))$ lies on the straight line between the origin and $(\frac{a}{b}, \frac{c}{d})$ (eg. $q = \frac{a}{2b}$, giving us $y(q) = y(\frac{a}{2b}) = \frac{c}{2d}$, note $(\frac{a}{2b}, \frac{c}{2d})$ lies exactly halfway between the origin and $(\frac{a}{b}, \frac{c}{d})$). Therefore $(\frac{a}{b}, \frac{c}{d})$ is not 'visible from the origin', so Euclid's orchard when generalised to the 2-dimensional rational lattice is a very empty orchard, possibly containing one rather lonely tree at the origin.

$\mathrm{EO}_n = \{(x_1, \ldots, x_n) \in \mathbb{P}^n : \gcd(x_1, \ldots, x_n) = 1\}$.

$\mathrm{GEO}_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : \gcd(x_1, \ldots, x_n) = 1\}$.

4 Symmetries

Check out [4].

5 Density

For each $a, b \in \mathbb{P}$ we shall denote $\{a, \ldots, b\} = [a, b] \cap \mathbb{Z}$ as ??.

For each $k \in \mathbb{P}$ let $d(k) = \frac{|\mathrm{GEO} \cap [0,k]^2|}{|[0,k]^2 \cap \mathbb{Z}^2|}$, ie. $d(k)$ is the proportion of points from $\{-k, \ldots, k\}^2$ that are contained in the generalised Euclid's orchard. We shall refer to $d(k)$ as the density of $\mathrm{GEO} \cap [0, k]$.

For each $k, n \in \mathbb{P}$ with $n \geq 2$, let $d_n(k) = \frac{|\mathrm{GEO}_n \cap [0,k]^n|}{|[0,k]^n \cap \mathbb{Z}^n|}$. We shall refer to $d_n(k)$ as the density of $\mathrm{GEO}_n \cap [0, k]^n$.

(probably don't need this) Note that due to symmetry, in 2 dimensions we can just consider the first octant (first 45 degrees of the first quadrant), ie consider $\{(x, y) \in \mathbb{N}^2 : y \leq x\}$.

We will be needing Euler's totient function $\varphi(n) = |\{a \in \{1, \ldots, n\} : gcd(a, n) = 1\}|$. (check out A000010 and A002088).

Also check out [1, Page 285, Theorem 332] and [2, Page 94]. Also check out [3, Pages 24-25] and [5] (cited on `Wikipedia`).

Proposition 5.1: $\lim_{k \to \infty} d(k) = \frac{6}{\pi^2}$.

James gave a proof in the James directory, also proved in lots of other places..

Code has given $\lim_{k \to \infty} d(k) = 0.607 = \frac{6}{\pi^2}$, $\lim_{k \to \infty} d_3(k) = 0.83$, $\lim_{k \to \infty} d_4(k) = 0.92$, $\lim_{k \to \infty} d_5(k) = 0.96$ and $\lim_{k \to \infty} d_6(k) = 0.98$.

6 Lattice Paths

We shall be considering integer sequences that arise when enumerating, possibly constrained, lattice paths using subsets of the generalised Euclid's orchard as step sets. Subsets to be used as step sets will include $\mathrm{GEO} \cap \mathbb{P}^2 = \mathrm{EO}$, $\mathrm{GEO} \cap \mathbb{N}^2$, $\mathrm{GEO} \cap (\mathbb{N} \times \mathbb{Z})$, $\mathrm{GEO} \cap \{(x, y) \in \mathbb{N}^2 : y \leq x\}$ and $\mathrm{GEO} \cap \{(x, y) \in \mathbb{N} \times \mathbb{Z} : |y| \leq x\}$. Subsets to be used as constraints will include $\mathbb{P}^2$, $\mathbb{N}^2$, $\{(x, y) \in \mathbb{N}^2 : y \leq x\}$, $\{(x, y) \in \mathbb{P}^2 : y \leq x\}$, $\{(x, y) \in \mathbb{N}^2 : y < x\}$, $\{(x, y) \in \mathbb{P}^2 : y < x\}$, $\{(x, y) \in \mathbb{N} \times \mathbb{Z} : |y| \leq x\}$ (and constraints involving lines of irrational slope maybe, if anything interesting appears), along with checking numerous other examples for any sequences that already appear on the OEIS (gold nuggets/diamonds). It would be amazing/awesome if any connections do appear for the sequences that will be examined in this paper.

Sequences will be considered along $y = x$, $y = 0$, numerous other lines along with the sequences formed for all nodes/vertices.

Roadmap:

(i) Lots of examples along with figures, maybe input (and maybe output) files for examples using existing code or independent code for enumerating more nodes/vertices than in the paper;

(ii) What combinations of finiteness properties and geometric conditions are possible?

(iii) Apply various results from JENH paper if/where possible.

7 Shortest Tours

It might be fun to play with shortest tours of finite subsets of Euclid's orchard. Starting point could be origin, for other starting positions the subset could be translated to be equivalent to a situation starting from the origin (possibly, maybe not for Euclid's orchard, as it might not be possible to get points where we need?). Maybe even consider non-crossing tours (do we then need to consider whether nodes are revisited?). What about shortest/longest paths to nodes? Path counts starting from and/or ending anywhere in a subset rather than from a particular point, and so on.

Including this section in this paper would probably require changing the paper title. Might be better to do separately, or come up with some other sections for recreational mathematics using (the generalised) Euclid's orchard.

8 The Origin

Similar to the question of whether the null graph is a pointless concept, see ?? (add citation), is the question of whether to include the origin in the generalised Euclid's orchard. Note that our definitions for GEO and GEO_n do not include the identity.

Arguments

(i) Intuitively whether one can see the origin may depend on whether one is allowed to look down;

(ii) $\gcd(0, \ldots, 0) = 0$ which suggests it should <u>not</u> be included;

(iii) Including the origin would be problematic when using (variations of) Euclid's orchard as a step set for lattice path enumeration (though could be excluded);

(iv) It will not affect symmetries;

(v) Including the origin would arguably be nicer for shortest paths starting from the origin, as the start of the path is then an element of Euclid's orchard as well.

(vi) Does it affect density in any way? (highly doubtful)

Acknowledgements

The author thanks... (what to write??)

References

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